Technical Comments

Comments on "An Analytic Solution for Entry into Planetary Atmospheres"

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RECENTLY Citron and Meir[2] obtained an analytic solution for entry into planetary atmospheres. Following Eggers,[3] they transformed the equations of entry dynamics into

$$(d^2f)/(dz^2) = (\alpha \cos^2\gamma/f)[\exp(z - z_e) - 1] - \xi \cos\gamma \quad (1)$$

$$\sin\gamma = \zeta(df/dz) \quad (2)$$

where $f = \rho/\rho_0$, $z = -\ln(V^2/rg)$, $\alpha = (\beta B^2)/(rg^2\rho_0^2)$, $\zeta = (\rho_0 g)/(\beta B)$, $\xi = (\beta BC_L)/(2g\rho_0 C_D)$, and $B = (mg)/(C_D A)$. To obtain an analytic solution, Citron and Meir[2] assumed that $\cos\gamma = \cos\gamma_e$ in Eq. (1) and expanded f in a power series in terms of the variable z, i.e., they assumed that

$$f = f_e + f_e'z + A_2z^2 + A_3z^3 + \ldots \quad (3)$$

$$f' = f_e' + 2A_2z + 3A_3z^2 + \ldots \quad (4)$$

Neglecting terms of higher order than z^2, they solved Eq. (4) for A_2 and substituted its value in Eq. (3) to obtain

$$f' + f_e' = (2/z)(f - f_e) \quad (5)$$

Multiplying Eq. (5) by f'' and integrating leads to

$$\frac{f'^2}{2} + f_e'f' - \frac{3}{2}f_e'^2 = \int_0^z \frac{2}{z} ff'' \left(1 - \frac{f_e}{f}\right) dz \quad (6)$$

They approximated the integral in Eq. (6) by neglecting "small terms." However, in the process of approximation, they neglected the term $(-ze^{-z_e})$ which comes from $\{-e^{-z_e} \times (f_e/f_e') \ln[1 + (f_e'/f_e)z]\}$ although they retained terms, which are of higher order than z, for example

$$e^{-z_e} \sum_{n=2}^{\infty} \frac{z^n}{nn!}$$

The importance of this neglected term depends on the value of f_e/f_e' compared to unity. In general, neglecting f_e/f_e' compared to unity is not reasonable because it can be very large. In physical variables,

$$f_e/f_e' = (\rho_e g)/(\beta B \sin\gamma_e) \quad (7)$$

Using, for illustration, the numerical values used by Citron and Meir[2] in obtaining Figs. 2 and 3, we find that f_e/f_e' is infinite because $\gamma_e = 0$.

Therefore, in general, the coefficient of z^2 in Eq. (49) of Citron and Meir[2] is incorrect, and, hence, their solution is correct only to order z rather than z^2 as they claimed. In that respect, the solution of Citron and Meir[2] has an order of error no less than the solution $(f = f_e + f_e'z)$ of Allen and Eggers[1] when f_e/f_e' is not very small.

Indeed, expanding the solution in power series of higher order than z will improve the solution of Allen and Eggers[1]: $f = f_e + f_e'z$. In order to obtain a solution correct to order z^3, we let $\cos\gamma$ in Eq. (1) be variable, substitute the series (3) in Eqs. (1) and (2), and equate the coefficients of equal powers of z to obtain

$$A_2 = (\{\alpha \cos^2\gamma_e[\exp(-z_e) - 1]\}/2f_e) - \tfrac{1}{2}\xi \cos\gamma_e \quad (8)$$

$$A_3 = \frac{\alpha \cos^2\gamma_e \exp(-z_e) - \xi f_e' \cos\gamma_e}{6f_e} + \left\{\frac{\xi\zeta^2 f_e}{\cos\gamma_e} - 2\alpha\zeta^2 [\exp(-z_e) - 1] - 1\right\}\left(\frac{f_e'A_2}{3f_e}\right) \quad (9)$$

The constants f_e and f_e' can be determined from the initial conditions. Once f is known, the altitude h can be determined from the isothermal condition $f = \exp(-\beta h)$, whereas γ can be determined from Eq. (2).

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References

[1] Allen, H. J. and Eggers, A. J., Jr., "A study of the motion and aerodynamic heating of ballistic missiles entering the earth's atmosphere at high supersonic speed," NACA Rept. 1381 (1958).

[2] Citron, S. J. and Meir, T. C., "An analytic solution for entry into planetary atmospheres," AIAA J. 3, 470–475 (1965).

[3] Eggers, A. J., Jr., "The possibility of a safe landing," *Space Technology* (John Wiley & Sons, Inc., New York, 1959), Chap. 13

Reply by Author to A. H. Nayfeh

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NAYFEH[1] makes two assertions in his comment. They are a) that a significant term of order $\bar{z}$ has been neglected in the approximate solution of Ref. 2, and hence the solution by Citron and Meier is no more accurate than that of Allen and Eggers $(f = f_e + f_e'\bar{z})$; and b) that expanding the solution in a power series, retaining higher order terms than $\bar{z}$, will usefully improve the solution of Allen and Eggers for small entry angles. In particular, Nayfeh develops the expansion through terms of order $\bar{z}^3$. Both of these assertions are incorrect.

Before proceeding to the analytical discussion, consider the example cited by Nayfeh to demonstrate his point. This is the case of tangential entry into the atmosphere at circular velocity, Figs. 2 and 3 of Ref. 2, where f_e/f_e' is infinite because $\gamma_e = 0$. The comparison given in Ref. 2† of the Citron-Meier solution for this case with the exact numerical solution is reproduced here in Figs. 1 and 2. In addition the results for the improved Allen and Eggers solution, developed in Ref. 1 by Nayfeh through terms of order $\bar{z}^3$, is also presented in Figs. 1 and 2.

Now, Nayfeh comments "the solution of Citron and Meir[2] has an order of error no less than the solution $(f = f_e + f_e'\bar{z})$ of Allen and Eggers[1] when f_e/f_e' is not very small." Yet, it is apparent from the results of Figs. 1 and 2 that the Citron-Meier solution is reasonably accurate, whereas the improved Allen and Eggers solution presented by Nayfeh is not. The reason for this failure of the series solution developed by Nayfeh is seen when the numerical values for the coefficients in

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† Equation (47) of Ref. 2 should read

$$f' = -\left(f_e' + \frac{J\bar{z}}{2}\right) + F(\bar{z})$$

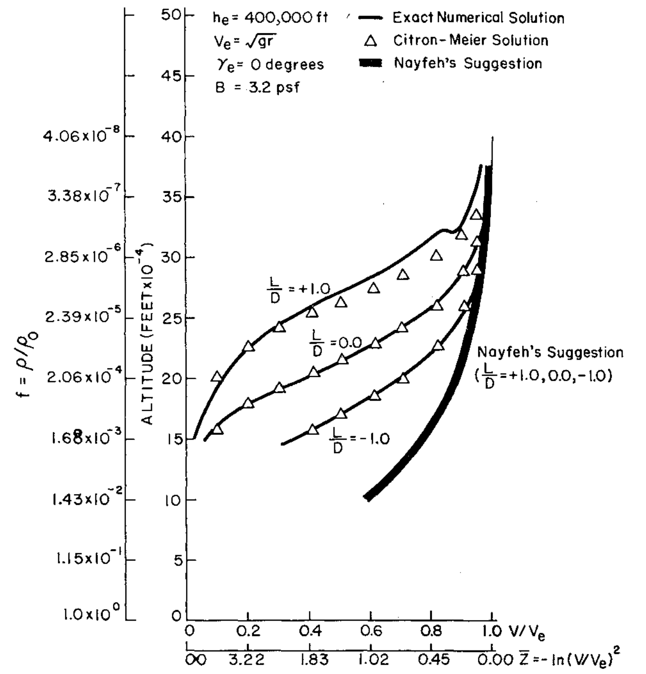

Fig. 1 Comparison of altitude profiles for tangential entry at circular satellite velocity. (ρ_0 = 0.0027 slugs/ft³, β^{-1} = 23,500 ft.).

the expansion are obtained. For example, for the case $L/D = -1$ in Figs. 1 and 2, using Eqs. 8 and 9 from Ref. 1, one has

$$f = 4.06 \times 10^{-8} + (0)\bar{z} + (0.39 \times 10^{-3})\bar{z}^2 + (0.11 \times 10^{-1})\bar{z}^3 \quad (1)$$

It is seen that convergence has been lost by the time $\bar{z} = 0.1$, ($V/V_e \simeq 0.95$).

Thus, whereas additional terms beyond those in the solution of Allen and Eggers may be retained (cf. Ref. 3),‡ the convergence difficulties of such a scheme for small entry angles limits its usefulness. It was just this difficulty that led the authors to seek the type of solution presented in Ref. 2.

Now, the following question could be raised. In the solution of Ref. 2, how can a term whose coefficient is f_e/f_e' be omitted when the value of f_e/f_e' approaches infinity and still, as demonstrated in Figs. 1 and 2, have an accurate solution result? Consider the coefficient in question which strictly should only be neglected when its value is very much less than unity. It is

$$f_e/f_e' = (\rho_e g)/(\beta B \sin\gamma_e)$$

Using the numerical values for earth ($\rho_e \simeq 10^{-10}$ slugs/ft³ at 400,000 ft, β^{-1} = 23,500 ft) one has

$$f_e/f_e' \simeq 10^{-2}/(B\gamma_e{}^\circ)$$

with B in pounds per square foot and γ_e in degrees. Even for a vehicle with values as small as $B = 1$ psf and $\gamma_e = 1°$, the term is only 1% of unity. Hence, in general, the term will be small compared to unity and can be neglected on that basis.

Alternatively, it is true that as γ_e approaches zero this term will grow without bound. However, the case $\gamma_e = 0$ only results in a nonskipping trajectory, the class of trajectories for which accuracy of the approximate solution of Ref. 2 was claimed, when the entry velocity is circular or subcircular. To obtain an interpretation of what has been done in neglecting the term to which Nayfeh refers, let us treat the case of tangential circular entry, as in Ref. 2 but now raking the entry density to be zero ($f_e = 0$).

In the notation of Ref. 2 the equation to be solved for circular entry ($\bar{z}_e = 0$) is

$$d^2f/d\bar{z}^2 = (I/f)(e^{\bar{z}} - 1) - J \quad (2)$$

where I and J are constants. The difficult term is the first one on the right. As in Ref. 2, to enable us to treat Eq. (2), the relation

$$f = [(f' + f_e')/2]\bar{z} \quad (3)$$

is obtained by considering a power series expansion of f in the variable $\bar{z}$ through terms of order $\bar{z}^2$. The entry density has been taken to be zero, hence, f_e does not appear in (3). Using (3) to eliminate f on the right hand of (2), one has, after multiplying both sides by $(f' + f_e')$, that

$$f''(f' + f_e') = (2I/\bar{z})(e^{\bar{z}} - 1) - J(f' + f_e') \quad (4)$$

Equation (4) now may be integrated directly to yield

$$\frac{f'^2}{2} + f_e'f' - \frac{3}{2}f_e'^2 = 2I\sum_{n=1}^{\infty}\frac{\bar{z}^n}{n\cdot n!} - J(f + f_e'\bar{z}) \quad (5)$$

Equation (3) then can be used in (5) to express f in the coefficient of J in terms of f'. Once this is done, the resulting relation is a quadratic equation in f', the solution of which yields

$$f' = -\left[f_e' + \frac{J\bar{z}}{2}\right] + 2F(\bar{z}) \quad (6)$$

where

$$F(\bar{z}) = \left[f_e'^2 - \frac{f_e'J\bar{z}}{2} + \left(\frac{J\bar{z}}{4}\right)^2 + I\sum_{n=1}^{\infty}\frac{\bar{z}^n}{n\cdot n!}\right]^{1/2} \quad (7)$$

On using (6) in (3), one also obtains

$$f = [-J\bar{z}^2/4] + \bar{z}F(\bar{z}) \quad (8)$$

Equations (6) and (8) with Eq. (7) defining $F(\bar{z})$, obtained here by neglecting f_e, are the same results§ obtained in

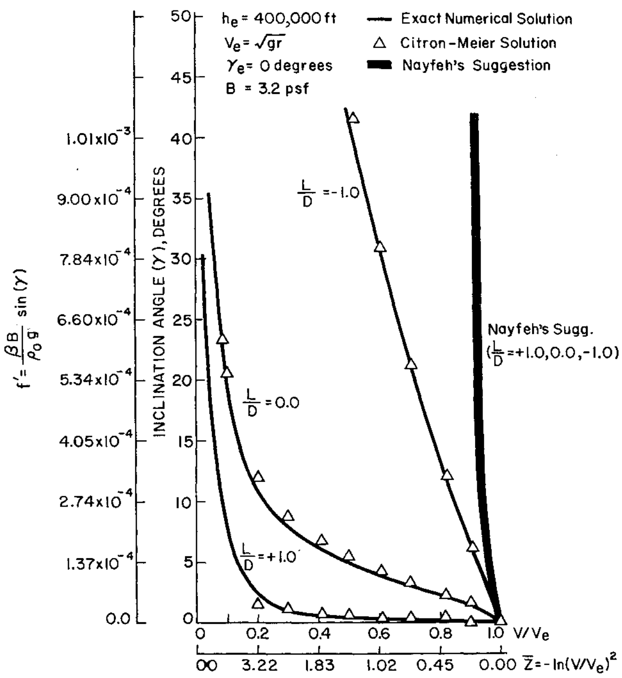

Fig. 2 Comparison of inclination angle variation for tangential entry at circular satellite velocity. (ρ_0 = 0.0027 slugs/ft³, β^{-1} = 23,500 ft.)

‡ Note that in the expansion by Eggers the entry density is taken as zero, producing a more convergent series than that obtained in Ref. 1 by Nayfeh.

§ Actually the left-hand side of (8) would have to be $(f - f_e)$ for the results to be identical, but f_e is so small that the difference is of no significance.

Ref. 2 for entry at circular velocity ($\bar{z}_e = 0$). The equivalence of the two solutions indicates that elimination of the term referred to by Nayfeh is based, in the limiting case $f_e' \to 0$, on restricting the effect that the small but nonzero entry density has on the solution.

Approximate solutions to physical problems must be judged on their accuracy and their utility. Figures 1 and 2 and the additional results of Ref. 2 provide a basis for assessing the value of the solution obtained in Ref. 2.

References

[1] Nayfeh, A. H., "Comments on 'An analytic solution for entry into planetary atmospheres'" AIAA J. **4**, 758 (1966).

[2] Citron, S. J. and Meier, T. C., "An analytic solution for entry into planetary atmospheres," AIAA J. **3**, 470–475 (1965).

[3] Eggers, A. J., Jr., "The possibility of a safe landing," *Space Technology* (John Wiley & Sons, Inc., New York, 1959), Chap. 13, pp. 13–18 and 13–19.

Comments on "Simplified Solutions for Ablation in a Finite Slab"

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IN a recent note, Chen[1] presents a solution of a differential equation, which is assumed to present a model of a charring ablator. The model used assumes that the original material is decomposed at a known ablation temperature T_m (all nomenclature used here are the same as those used by Chen) and a known surface temperature T_0. The latter assumption does not recognize the fact that the solution of the problem of charring ablation is governed by heat balance rather than by an imposed surface temperature. The net heat transferred to the material must equal the heat stored plus the heat absorbed by decomposition. Specification of the surface temperature presupposes the knowledge of the heat balance, which can be obtained only after the complete problem is solved. The surface boundary condition for negligible radiation therefore should be written as: heated conducted into the material = aerodynamic heat input − heat blocked by injection of gases into the boundary layer. Neither the net heat transfer into the material nor the surface temperature is known a priori.

This, however, is a minor point in comparison with the assumption implied in Chen's basic equation [Eq. (1)]. The equation neglects the convective effects, i.e., the heat transferred from the char to the gaseous products of decomposition. A simple order of magnitude analysis yields the ratio

$$\frac{\text{Heat absorbed by the gases}}{\text{Heat absorbed by the decomposition}} \approx \frac{(T_0 - T_m)}{L} C_{pg} \frac{(\rho_u - \rho_c)}{\rho_u}$$

where C_{pg} = specific heat of the gases. In any practical application of any common charring ablator for which a reaction temperature is assumed, this ratio is about one or greater. There are indications that some materials have a very low heat of depolymerization, so that the ratio actually can be much greater than one. Thus, Chen's solution neglects an effect that is as important as the one that is retained in his boundary condition.

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It seems, therefore, that Chen's simplifying assumptions lead to a steady-state solution for a constant temperature sink (reaction zone) with known surface heat transfer and surface temperature, rather than for a charring ablator.

Reference

[1] Chen, N. H., "Simplified solutions for ablation in a finite slab," AIAA J. **3**, 1148 (1965).

Reply by Author to A. Wortman

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ABLATION is a complicated phenomenon. It deals with conduction, diffusion, chemical reaction (or combustion), kinetics, etc. The products of the combustion are char and multicomponent gaseous mixtures. In addition, the heat input from the environment to initiate the ablation in a rocket motor also is concerned with multicomponent exhaust gaseous mixtures. Because of this complex nature, it is certainly impossible to obtain a closed form solution to account for all these effects. Hence, in order to have an approximate analytical expression, "simplified assumptions" must be made.

With these simplified assumptions in mind, the following "heat" terms, as indicated by A. Wortman, already have been considered in my recent note.[1] The heat blocked by injection of gases into the boundary layer was eliminated because of assumption 7 in my note.[1] The heat absorbed by the gases from the decomposition was neglected on account of assumption 3.[1] The heat absorbed by the gases from the environment was taken into consideration in the char-gas layer.

In regard to the surface temperature T_0, it is true that it varies with time. However, for a particular instant, there exists such a temperature. This temperature is not known, as indicated by A. Wortman, but is determined from the heat balance equations in my note.[1]

Moreover, T_m was defined as the ablation temperature in my note, but not as the reaction temperature. These two temperatures may not be equal. The latter one, as suggested by A. Wortman, does not fit into my proposed model.

Reference

[1] Chen, N. H., "Simplified solutions for ablation in a finite slab," AIAA J. **3**, 1148–1149 (1965).

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Comments on "Calibration of Preston Tubes in Supersonic Flow"

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SIGALLA[1] has noted an interesting application of the reference-temperature method for compressible boundary layers to a calibration formula for Preston tubes. The writers

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